

# DEL OPERATOR

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The del operator, written  $\nabla$ , is the vector differential operator. In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x} \mathbf{a}_x + \frac{\partial}{\partial y} \mathbf{a}_y + \frac{\partial}{\partial z} \mathbf{a}_z$$

in cylindrical coordinates as

$$\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$$

in spherical coordinates:

$$\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$



# GRADIENT OF A SCALAR

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The **gradient** of a scalar field  $V$  is a vector that represents both the magnitude and the direction of the maximum space rate of increase of  $V$ .

the gradient of  $V$  can be expressed in Cartesian, cylindrical, and spherical coordinates.

For Cartesian co ordinates

$$\nabla V = \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z$$

for cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z$$

and for spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi$$



**EXAMPLE**

Find the gradient of the following scalar fields:

(a)  $V = e^{-z} \sin 2x \cosh y$

(b)  $U = \rho^2 z \cos 2\phi$

(c)  $W = 10r \sin^2 \theta \cos \phi$

**Solution:**

$$\begin{aligned} \text{(a) } \nabla V &= \frac{\partial V}{\partial x} \mathbf{a}_x + \frac{\partial V}{\partial y} \mathbf{a}_y + \frac{\partial V}{\partial z} \mathbf{a}_z \\ &= 2e^{-z} \cos 2x \cosh y \mathbf{a}_x + e^{-z} \sin 2x \sinh y \mathbf{a}_y - e^{-z} \sin 2x \cosh y \mathbf{a}_z \end{aligned}$$

$$\begin{aligned} \text{(b) } \nabla U &= \frac{\partial U}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial U}{\partial \phi} \mathbf{a}_\phi + \frac{\partial U}{\partial z} \mathbf{a}_z \\ &= 2\rho z \cos 2\phi \mathbf{a}_\rho - 2\rho z \sin 2\phi \mathbf{a}_\phi + \rho^2 \cos 2\phi \mathbf{a}_z \end{aligned}$$

$$\begin{aligned} \text{(c) } \nabla W &= \frac{\partial W}{\partial r} \mathbf{a}_r + \frac{1}{r} \frac{\partial W}{\partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi} \mathbf{a}_\phi \\ &= 10 \sin^2 \theta \cos \phi \mathbf{a}_r + 10 \sin 2\theta \cos \phi \mathbf{a}_\theta - 10 \sin \theta \sin \phi \mathbf{a}_\phi \end{aligned}$$

# DIVERGENCE OF A VECTOR

The **divergence** of  $\mathbf{A}$  at a given point  $P$  is the *outward* flux per unit volume as the volume shrinks about  $P$ .

the divergence of  $\mathbf{A}$  at point  $P(x_0, y_0, z_0)$  in a Cartesian system is given by

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

In cylindrical coordinates.

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

the divergence of  $\mathbf{A}$  in spherical coordinates as

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$



**EXAMPLE**

Determine the divergence of these vector fields:

(a)  $\mathbf{P} = x^2yz \mathbf{a}_x + xz \mathbf{a}_z$

(b)  $\mathbf{Q} = \rho \sin \phi \mathbf{a}_\rho + \rho^2 z \mathbf{a}_\phi + z \cos \phi \mathbf{a}_z$

(c)  $\mathbf{T} = \frac{1}{r^2} \cos \theta \mathbf{a}_r + r \sin \theta \cos \phi \mathbf{a}_\theta + \cos \theta \mathbf{a}_\phi$

**Solution:**

$$\begin{aligned} \text{(a)} \quad \nabla \cdot \mathbf{P} &= \frac{\partial}{\partial x} P_x + \frac{\partial}{\partial y} P_y + \frac{\partial}{\partial z} P_z \\ &= \frac{\partial}{\partial x} (x^2yz) + \frac{\partial}{\partial y} (0) + \frac{\partial}{\partial z} (xz) \\ &= 2xyz + x \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \nabla \cdot \mathbf{Q} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho Q_\rho) + \frac{1}{\rho} \frac{\partial}{\partial \phi} Q_\phi + \frac{\partial}{\partial z} Q_z \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 \sin \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (\rho^2 z) + \frac{\partial}{\partial z} (z \cos \phi) \\ &= 2 \sin \phi + \cos \phi \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \nabla \cdot \mathbf{T} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (T_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (T_\phi) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (\cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin^2 \theta \cos \phi) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\cos \theta) \\ &= 0 + \frac{1}{r \sin \theta} 2r \sin \theta \cos \theta \cos \phi + 0 \\ &= 2 \cos \theta \cos \phi \end{aligned}$$

